Written Exam Economics Summer 2020

Advanced Financial and Macro Econometrics

Comments on Solution

1 Bootstrap

#1.1 Time Series with Missing Observations

(a) It follows that

$$y_t = \rho^2 y_{t-2} + (\varepsilon_t + \rho\varepsilon_{t-1}) \tag{1.1}$$

and hence

$$x_n = \phi x_{n-1} + \eta_n \tag{1.2}$$

with $\phi = \rho^2$ and η_n i.i.d.$\mathrm{N}\left(0, \sigma_\eta^2\right)$ with $\sigma_\eta^2 = (1+\rho^2)\,\sigma_\varepsilon^2$.

#1.2 Likelihood Ratio Test

(a) By definition under H,

$$\hat{\phi}_N - f = \frac{\sum_{n=1}^{N} \eta_n x_{n-1}}{\sum_{n=1}^{N} x_{n-1}^2}, \tag{1.3}$$

and the result follows by applying the CLT to the MGD $m_n = \eta_n x_{n-1}$ (argue that it is a MGD) and LLN to $N^{-1}\sum_{n=1}^{N} x_{n-1}^2$. Arguments to be included: verification of CLT conditions, MGD, and that as $|f| < 1$ then in particular x_n is stationary with finite $\mathrm{E}(x_n^2) = \sigma_\eta^2 / \left(1 - f^2\right)$.

#1.3 Wild Bootstrap Scheme

(a) Distribution of x_n^* conditional on x_{n-1}^* and the original data: $N\left(f x_{n-1}^*, \hat{\eta}_n^2\right)$
As $x_n^* = f^n x_0 + \sum_{i=0}^{n-1} f^i \eta_n^*$ and $\mathrm{E}^*\left(x_n^*\right) = f^n x_0 = 0$ if $x_0 = 0$.

#1.4 Bootstrap Statistic

(a) m_n^* is a MGD sequence wrt $\mathcal{F}_n^* = \sigma\left(x_n^*, x_{n-1}^*, ..., x_0^*\right)$ as $\mathrm{E}^*\left(m_n^* | \mathcal{F}_{n-1}^*\right) = x_{n-1}^*\,\mathrm{E}^*\left(\eta_n^*\right) = 0$.

(*b*) Follows by applying the CLT for MGD sequences by noting that:

$$\mathrm{E}^*\left(m_n^{*2}|\mathcal{F}_{n-1}^*\right) = x_{n-1}^{*2}\hat{\eta}_n^2 = x_{n-1}^{*2}\left(\eta_n^2 - \sigma_\eta^2\right) + x_{n-1}^{*2}\sigma_\eta^2, \tag{1.4}$$

such that in particular

$$\begin{aligned} N^{-1}\sum_{n=1}^{N} x_{n-1}^{*2}\hat{\eta}_n^2 &= \sigma_\eta^2 N^{-1}\sum_{n=1}^{N} x_{n-1}^{*2} + N^{-1}\sum_{n=1}^{N} x_{n-1}^{*2}\left(\hat{\eta}_n^2 - \sigma_\eta^2\right) \\ &\overset{P^*}{\to}_P \sigma_\eta^4/\left(1-f^2\right). \end{aligned} \tag{1.5}$$

(*c*) That $W_N^* \overset{D^*}{\to}_P \chi_1^2$ under H, follows as for the non-bootstrap case by Question 1.2(a).

2 Co-integration

#2.1 Pairs-Trading

(*a*) The postulated co-integration structure,

$$\beta = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

implies that all spreads, e.g. $A-B$, $A-C$, $A-D$, $B-C$, $B-D$, $C-D$ are stationary and candidates for pairs-trading as described in the lecture note. The investor could look for large deviations from equilibrium and buy and sell the relevant pair of stocks.
More generally, the investor could also look at linear combinations of the spreads, and find some $b \in \mathrm{span}(\beta)$ with a large deviation from equilibrium.

#2.2 Missing Observations

(*a*) It holds that

$$\begin{aligned} Y_t &= (I_p+\Pi)Y_{t-1} + \epsilon_t \\ &= (I_p+\Pi)\left((I_p+\Pi)Y_{t-2} + \epsilon_{t-1}\right) + \epsilon_t \\ &= (I_p+\Pi)^2 Y_{t-2} + \epsilon_t + (I_p+\Pi)\epsilon_{t-1}, \end{aligned} \tag{2.1}$$

or for $n = 1, 2, ..., N$, with $N = T/2$ an integer,

$$X_n = (I_p+\Pi)^2 X_{n-1} + \eta_n, \tag{2.2}$$

with $\{\eta_n\}_{n=1}^N$ related to $\{\epsilon_t\}_{t=1}^T$ as

$$\eta_n = \epsilon_{2n} + (I_p + \Pi)\epsilon_{2n-1}. \tag{2.3}$$

Observe that $\{\eta_n\}_{n=1}^N$ is an i.i.d. sequence. The sequence $\epsilon_t + (I_p+\Pi)\epsilon_{t-1}$ is MA(1), but we only observe every second observation and

$$\mathrm{corr}(\eta_n, \eta_{n-1}) = \mathrm{corr}(\epsilon_{2n} + (I_p + \Pi)\epsilon_{2n-1}, \epsilon_{2n-2} + (I_p + \Pi)\epsilon_{2n-3}) = 0. \tag{2.4}$$

For the error-correction form, with $\Delta X_n = X_n - X_{n-1}$, we get

$$\Delta X_n = P X_{n-1} + \eta_n, \tag{2.5}$$

with

$$P = (I_p + \Pi)^2 - I_p = \Pi^2 + 2\Pi = \alpha\beta'\alpha\beta' + 2\alpha\beta' = \alpha(\beta'\alpha + 2I_r)\beta' = \alpha\theta\beta'. \tag{2.6}$$

(*b*) If the process Y_t is assumed to be I(1) and co-integrated, i.e. Assumption 3.1 in the lecture note, it holds that $\beta'\alpha + I_r$ has all eigenvalues inside the unit circle, such that the matrix $(\beta'\alpha + 2I_r)$ has full rank r and it follows that the co-integration rank of X_n is $r_X = \mathrm{rank}(P) = \mathrm{rank}(\Pi) = r$. And with $P = ab'$ we can choose

$$b = \beta \quad \text{and} \quad a = \alpha\theta, \tag{2.7}$$

such that the equilibrium relationships are unchanged while the error correction is faster.

(*c*) The moving average solution is similar to the usual co-integrated VAR. The stochastic trends are given by

$$\tau_n = \sum_{i=1}^{n} a_\perp' \eta_i. \tag{2.8}$$

Note that we can choose $a_\perp = \alpha_\perp$ because

$$a_\perp' a = \alpha_\perp' \alpha\theta = 0 \quad \text{and} \quad a' a_\perp = \theta'\alpha\alpha_\perp = 0. \tag{2.9}$$

It is the same linear combinations of η that have permanent effects on X as the linear combinations of ϵ with permanent effects on Y. Also recall that $\eta_n = \epsilon_{2n} + (I_p + \Pi)\epsilon_{2n-1}$.

The loading to the common trend are given by

$$b_\perp(a_\perp' b_\perp)^{-1} = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}, \tag{2.10}$$

which are unchanged.

#2.3 Empirical Analyses

(*a*) Both series should be modelled.

[1] The series can be modelled without deterministics or with a restricted or unrestricted constant. In any case results are very similar and typically only one lag is needed to capture the dynamics.

[2] The co-integration rank is typically found to be three as suggested by the theoretical introduction.

In the present case the sample is fairly large ($T = 400$ and $N = 200$) and there are no signs of heteroskedasticity in which case the asymptotic approximation of the rank test is expected to be accurate. For the present case, the result of the bootstrap test will be quite close to the result of the asymptotic test.

[3] With the rank $r = 3$ imposed, the suggested structure for β is typically not rejected. For the case with a restricted constant and for the weekly data, we find

```
beta', the normalized cointegrating vectors:
              Aw_1        Bw_1        Cw_1        Dw_1    Constant
  CVec(1)        1          -1           0           0     -0.0467
                                                          {  -1.8}
  CVec(2)        1           0          -1           0      0.0681
                                                          {   2.7}
  CVec(3)        1           0           0          -1      0.0331
{t-value}                                                 {   1.1}

alpha, the loadings on the cointegrating vectors:
          alpha[][0]  alpha[][1]  alpha[][2]
      DAw     -0.225     -0.0733      -0.157
            {  -7.0}    {  -2.5}    {  -5.1}
      DBw     0.0544     -0.0338      0.0223
            {   1.7}    {  -1.1}    {   0.7}
      DCw    -0.0662       0.151      0.0319
            {  -2.0}    {   4.9}    {   1.0}
      DDw    -0.0489     -0.0149      0.0888
{t-value}   {  -1.6}    {  -0.5}    {   3.1}
```

while the bi-weekly data yields

```
beta', the normalized cointegrating vectors:
              Abw_1       Bbw_1       Cbw_1       Dbw_1    Constant
  CVec(1)         1          -1           0           0     -0.0496
                                                           {  -2.0}
  CVec(2)         1           0          -1           0      0.0694
                                                           {   2.8}
  CVec(3)         1           0           0          -1      0.0399
{t-value}                                                  {   1.2}

alpha, the loadings on the cointegrating vectors:
          alpha[][0]  alpha[][1]  alpha[][2]
     DAbw     -0.389      -0.124      -0.224
            {  -6.5}    {  -2.2}    {  -3.9}
     DBbw      0.109     -0.0446      0.0703
            {   1.7}    {  -0.7}    {   1.1}
     DCbw     -0.185       0.263      0.0143
            {  -2.9}    {   4.4}    {   0.2}
     DDbw     -0.139     -0.0184       0.139
{t-value}   {  -2.4}    {  -0.3}    {   2.5}
```

[4] For the MA representation, we find for the restricted system that $\beta_{\perp} =$

$(1,1,1,1)'$ while the estimated $\alpha_\perp$'s are given by (weekly data):

```
alpha_ort', alpha orthogonal (transposed):
                     Aw          Bw          Cw          Dw
         CT1      0.167           1       0.299     -0.0624
   {t-value}     {    1.1}               {    1.3}   {  -0.2}
```

and bi-weekly data:

```
alpha_ort', alpha orthogonal (transposed):
                    Abw         Bbw         Cbw         Dbw
         CT1       0.22           1       0.261      -0.178
   {t-value}     {    1.2}               {    1.0}   {  -0.5}
```

(*b*) Qualitatively, the two sets of results are very similar. We find the same structure for β and for the weekly data we find an error correction given by

$$\hat{\alpha} = \begin{pmatrix} -0.225 & -0.0733 & -0.157 \\ 0.0544 & -0.0338 & 0.0223 \\ -0.0662 & 0.151 & 0.0319 \\ -0.0489 & -0.0149 & 0.0888 \end{pmatrix}. \tag{2.11}$$

The implied bi-weekly error-correction, a, is given by

$$a = \hat{\alpha}(\hat{\beta}'\hat{\alpha} + 2I_r) = \begin{pmatrix} -0.348 & -0.112 & -0.221 \\ 0.0950 & -0.0635 & 0.0358 \\ -0.144 & 0.269 & 0.0393 \\ -0.0974 & -0.0297 & 0.167 \end{pmatrix}, \tag{2.12}$$

which is quite close to the estimated a for the bi-weekly data

$$\hat{a} = \begin{pmatrix} -0.389 & -0.124 & -0.224 \\ 0.109 & -0.0446 & 0.0703 \\ -0.185 & 0.263 & 0.0143 \\ -0.139 & -0.0184 & 0.139 \end{pmatrix}. \tag{2.13}$$

(*c*) The deviation from the three equilibrium relationships are given by

Some promising pairs-trading opportunities include the second spread in the beginning of the period or the third spread around observation 30 or 150.